

NON-ADDITIVE FUNCTORS AND EULER CHARACTERISTICS

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ABSTRACT. We show under suitable finiteness conditions that a functor between abelian categories induces a (not necessarily additive) map between their Grothendieck groups. This is related to the derived functors of Dold and Puppe, and generalizes a theorem of Dold.

1. INTRODUCTION

Let \mathcal{A} and \mathcal{B} be abelian categories. Assume that \mathcal{A} has enough projectives. If $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor, not necessarily additive, then we denote by

$$LF: \mathcal{K}_{\geq 0}(\mathcal{A}) \rightarrow \mathcal{K}_{\geq 0}(\mathcal{B}),$$

the total derived functor of Dold and Puppe [3], where $\mathcal{K}_{\geq 0}$ is the homotopy category of chain complexes concentrated in non-negative degrees. The definition of LF involves taking a projective resolution, using the Dold-Kan theorem to pass to an associated simplicial object, applying F and taking the associated chain complex. The details will be recalled in Section 5. If F is additive, then LF coincides with the usual total left derived functor. Note that our notation is slightly non-standard, in that we denote by LF the functor between homotopy categories, instead of the induced functor between derived categories.

We will require that $F(0) = 0$ and that F is of *degree* $\leq d$ for some positive integer d . The definition is recursive. F is said to be of degree ≤ 1 if it is additive, and of degree $\leq d$ if and only if there exists a functor $G: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ of degree $\leq d - 1$ in both arguments, together with a functorial decomposition

$$F(X \oplus Y) = F(X) \oplus F(Y) \oplus G(X, Y).$$

Examples of functors of degree $\leq d$ are the Schur functors \wedge^d and Sym^d .

Let \mathcal{A}_0 be a weak Serre subcategory of \mathcal{A} (this is a slight generalization of the more common notion of a Serre subcategory, see [10, Tag 02MN]). Let $\mathcal{K}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A})$ be the full subcategory of $\mathcal{K}_{\geq 0}(\mathcal{A})$ consisting of those complexes X_{\bullet} such that

- (1) $H_i(X_{\bullet}) = 0$ for $i \gg 0$;
- (2) $H_i(X_{\bullet}) \in \mathcal{A}_0$ for all i .

Every object $X_{\bullet} \in \mathcal{K}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A})$ has an Euler characteristic

$$\chi(X_{\bullet}) = \sum_i (-1)^i [H_i(X_{\bullet})]$$

in the Grothendieck group $K_0(\mathcal{A}_0)$. Let \mathcal{B}_0 be a weak Serre subcategory of \mathcal{B} , and define $\mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$ analogously.

Our main result is the following theorem.

Key words and phrases. Grothendieck groups, non-additive functors.

Theorem 1. *Assume that $F: \mathcal{A} \rightarrow \mathcal{B}$ is of finite degree $\leq d$ with $F(0) = 0$, and that LF maps $\mathcal{K}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A})$ to $\mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$. Then there is a unique map $f: K_0(\mathcal{A}_0) \rightarrow K_0(\mathcal{B}_0)$ such that the square*

$$(1) \quad \begin{array}{ccc} \mathcal{K}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A}) & \xrightarrow{LF} & \mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B}) \\ \downarrow \chi & & \downarrow \chi \\ K_0(\mathcal{A}_0) & \xrightarrow{f} & K_0(\mathcal{B}_0) \end{array}$$

commutes. Moreover, the function f is of degree $\leq d$.

Here f being of degree $\leq d$ is also defined recursively: f is of degree ≤ 1 if it is linear, and of degree $\leq d$ if the function

$$g(x, y) := f(x + y) - f(x) - f(y)$$

is of degree $\leq d - 1$ in both arguments.

Typically, $K_0(\mathcal{A}_0)$ is only interesting if \mathcal{A}_0 is small enough, whereas \mathcal{A} needs to be large enough to contain projective resolutions (or with the dual version of the theorem, injective resolutions) of objects in \mathcal{A}_0 . For example \mathcal{A}_0 could be the category of finite abelian groups, or of coherent \mathcal{O}_X -modules on a scheme X , with \mathcal{A} the category of finitely generated abelian groups respectively quasi-coherent \mathcal{O}_X -modules. In both cases the natural map $K_0(\mathcal{A}_0) \rightarrow K_0(\mathcal{A})$ is the zero map.

If $\mathcal{A}_0 = \mathcal{A}$ and if every object of \mathcal{A} has a finite projective resolution, then Theorem 1 is a theorem of Dold [2]. Our proof is similar in flavour to Dold's, but because we cannot just compute Euler characteristics in $K_0(\mathcal{A})$, we need more refined constructions that only involve chain complexes (or simplicial objects) in \mathcal{A} with homology in \mathcal{A}_0 . A crucial ingredient in our proof is a double complex due to Köck [8], which induces a functorial resolution of $F(P/Q)$ for a quotient P/Q in terms of cross effects of F applied to P and Q , see §3.

We end this introduction with three examples where we compute the map f of the theorem explicitly.

Example 1. If $F: \mathcal{A} \rightarrow \mathcal{B}$ is *additive* then the theorem follows immediately from the long exact sequence of homology. The map f is additive, and characterized by

$$f([X]) = \sum_i (-1)^i [L_i F X]$$

in $K_0(\mathcal{B}_0)$, for all $X \in \mathcal{A}_0$. (Note that the sum is finite because of our assumption on LF).

Example 2. Let R be a commutative ring and let $\mathcal{A} = \mathcal{A}_0$ be the category of finitely generated R -modules. Assume every $M \in \mathcal{A}$ has a finite projective resolution. Consider the symmetric power $F = \text{Sym}^d: \mathcal{A} \rightarrow \mathcal{A}$. Every element of $K_0(\mathcal{A})$ is of the form $[P] - [Q]$ for projective objects P, Q in \mathcal{A} . Consider a complex $Q \xrightarrow{0} P$ in degree 1 and 0. Using a theorem of Quillen [6, I.4.3.2] one can show that

$$L_i \text{Sym}^d(Q \rightarrow P) = \text{Sym}^{d-i} P \otimes \wedge^i Q$$

for every i (See also [8, 2.4]). It follows that the map f of Theorem 1 is

$$f: K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A}), [P] - [Q] \mapsto \sum_{i=0}^d (-1)^i [\text{Sym}^{d-i} P \otimes \wedge^i Q],$$

and in particular that this map is well-defined. Similarly, one finds for $G = \wedge^d$ the map

$$g: K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A}), [P] - [Q] \mapsto \sum_{i=0}^d (-1)^i [\wedge^{d-i} P \otimes \Gamma^i Q]$$

where $\Gamma^i Q$ denotes the i -th divided power module of Q .

In the third example we consider a situation where it is necessary to separate the roles of \mathcal{A} and \mathcal{A}_0 .

Example 3. Let $\mathcal{A}_0 = \mathcal{B}_0$ be the category of finite (torsion) \mathbf{Z} -modules, and $\mathcal{A} = \mathcal{B}$ the category of all \mathbf{Z} -modules. Consider the functors $F = \text{Sym}^d$ and $G = \wedge^d$ from \mathcal{A} to \mathcal{B} . If X_\bullet is in $\mathcal{K}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A})$ then the groups $L_i F(X_\bullet)$ and $L_i G(X_\bullet)$ are finitely generated and vanish for $i \gg 0$. Moreover, they vanish after tensoring with \mathbf{Q} , hence $LF(X_\bullet)$ and $LG(X_\bullet)$ lie in $\mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$.

The cardinality of a module defines an isomorphism $K_0(\mathcal{A}_0) \cong \mathbf{Q}_{>0}^\times$. We claim that the induced maps $f, g: \mathbf{Q}_{>0}^\times \rightarrow \mathbf{Q}_{>0}^\times$ are given by

$$f(x) = x \quad \text{and} \quad g(x) = x^{(-1)^{d-1}},$$

for all $x \in \mathbf{Q}^\times$.

Indeed: since f and g are of degree $\leq d$, it suffices to show the above identities for all positive integers x . Let m be a positive integer and consider the complex X consisting of a cyclic group $\mathbf{Z}/m\mathbf{Z}$ placed in degree 0. Because $\wedge^i \mathbf{Z} = 0$ for $i \neq 0, 1$ one can easily compute $L_i \text{Sym}^d X$ and $L_i \wedge^d X$ using the Koszul complexes associated to the resolution $\mathbf{Z} \xrightarrow{m} \mathbf{Z}$ of X (see [8, 2.4, 2.7]). One finds

$$L_i \text{Sym}^d(\mathbf{Z}/m\mathbf{Z}) \cong \begin{cases} \mathbf{Z}/m\mathbf{Z} & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$

and

$$L_i \wedge^d(\mathbf{Z}/m\mathbf{Z}) \cong \begin{cases} 0 & (i \neq d-1) \\ \mathbf{Z}/m\mathbf{Z} & (i = d-1). \end{cases}$$

and hence $f(m) = m$ and $g(m) = m^{(-1)^{d-1}}$ as claimed. (For an alternative computation of $L_i \text{Sym}^d(\mathbf{Z}/m\mathbf{Z})$ and $L_i \wedge^d(\mathbf{Z}/m\mathbf{Z})$ see Jean [7, §2.3].)

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2. CROSS-EFFECTS AND FUNCTORS OF FINITE DEGREE

In this section we briefly summarize the definition and main properties of cross effect functors. We refer to the original text of Eilenberg and Mac Lane [4] for proofs and more details.

Let \mathcal{A} be an abelian category. Let $X_1, \dots, X_n \in \mathcal{A}$. Put $X := X_1 \oplus \dots \oplus X_n$. Let $e_i \in \text{End } X$ be the idempotent with image X_i . For a subset $S \subset I = \{1, \dots, n\}$

we define

$$e_S := \sum_{i \in S} e_i \in \text{End } X.$$

This is the idempotent with image $\oplus_{i \in S} X_i \subset X$.

Now let \mathcal{B} be an abelian category and let F be a functor $\mathcal{A} \rightarrow \mathcal{B}$ with $F(0) = 0$. Consider the endomorphism

$$\text{cr}_n(e_1, \dots, e_n) := \sum_{S \subset I} (-1)^{n-|S|} F(e_S) \in \text{End } F(X).$$

Then one verifies that $\text{cr}_n(e_1, \dots, e_n)$ is *idempotent*.

Definition 1. The n -th *cross effect* of F is the functor $F_n: \mathcal{A}^n \rightarrow \mathcal{B}$ given by

$$F_n(X_1, \dots, X_n) := \text{im } \text{cr}_n(e_1, \dots, e_n) \subset F(X_1 \oplus \dots \oplus X_n).$$

Note that $F_0 = 0$ and $F_1 = F$. We now list some more basic properties that are useful in working with the cross effect functors.

Proposition 1. *There are isomorphisms*

$$F(X_1 \oplus X_2) = F(X_1) \oplus F(X_2) \oplus F_2(X_1, X_2),$$

functorial in X_1 and X_2 . The functor F is additive if and only if F_2 vanishes. \square

The higher cross-effect functors satisfy a kind of associativity property that allows them to be computed in a recursive way:

Proposition 2. *Let $X_1, \dots, X_n \in \mathcal{A}$. Let $G: \mathcal{A} \rightarrow \mathcal{B}$ be the functor given by*

$$G(Y) := F_{n+1}(X_1, \dots, X_n, Y)$$

then there are isomorphisms

$$G_m(Y_1, \dots, Y_m) = F_{m+n}(X_1, \dots, X_n, Y_1, \dots, Y_m),$$

functorial in the X_i and Y_i . \square

Propositions 1 and 2 give a decomposition

$$F(X_1 \oplus \dots \oplus X_n) = \bigoplus_{0 < d \leq n} \bigoplus_{i_1 < \dots < i_d} F_d(X_{i_1}, \dots, X_{i_d}).$$

Definition 2. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor with $F(0) = 0$. Let d be a positive integer. We say that F is of *degree* $\leq d$ if the functor F_{d+1} vanishes.

Using Propositions 1 and 2 one sees that this definition coincides with the recursive definition given in the introduction.

3. KÖCK'S RESOLUTION

Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor with $F(0) = 0$. Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be a split short exact sequence in \mathcal{A} . Following Kock [8], we will describe an explicit resolution of $F(Z)$ in terms of the map $X \rightarrow Y$ and the cross effect functors of F . We do not choose a preferred splitting, and insist that all constructions be functorial in the short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$.

Let $n > 0$. For $1 \leq i \leq n$ consider the maps $\delta_{n,i}: X^{\oplus(n+1)} \rightarrow X^{\oplus n}$ given by

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_{n+1})$$

These induce maps

$$F(\delta_{n,i}): F(X^{\oplus(n+1)}) \rightarrow F(X^{\oplus n})$$

which restrict to maps

$$F(\delta_{n,i}): F_{n+1}(X, \dots, X, X) \rightarrow F_n(X, \dots, X).$$

Let d_n be the map

$$d_n = \sum_i (-1)^i F(\delta_{n,i}): F_{n+1}(X, \dots, X, X) \rightarrow F_n(X, \dots, X).$$

One verifies directly that

$$\cdots \longrightarrow F_3(X, X, X) \xrightarrow{d_2} F_2(X, X) \xrightarrow{d_1} F(X)$$

forms a complex in \mathcal{B} . Using the map $X \rightarrow Y$, one similarly constructs a complex

$$\cdots \longrightarrow F_3(X, X, Y) \longrightarrow F_2(X, Y) \longrightarrow F(Y).$$

Let C be the double chain complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_3(X, X, Y) & \longrightarrow & F_2(X, Y) & \longrightarrow & F(Y) \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F_3(X, X, X) & \longrightarrow & F_2(X, X) & \longrightarrow & F(X) \end{array}$$

with $F(Y)$ in degree $(0,0)$. The map $Y \rightarrow Z$ induces a map $F(Y) \rightarrow F(Z)$ and one obtains an augmented double complex

$$(2) \quad \begin{array}{cccccccc} \cdots & \longrightarrow & F_3(X, X, Y) & \longrightarrow & F_2(X, Y) & \longrightarrow & F(Y) & \longrightarrow & F(Z) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F_3(X, X, X) & \longrightarrow & F_2(X, X) & \longrightarrow & F(X) & \longrightarrow & 0 \end{array}$$

functorial in the short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$.

Theorem 2. *The map $\text{Tot } C \rightarrow F(Z)$ induced by (2) is a quasi-isomorphism.*

This follows from [8, Lemma 2.2], where more generally it is shown that for every map $X \rightarrow Y$ the complex $\text{Tot } C$ computes the total derived functor of F applied to $X \rightarrow Y$ (at least if X and Y are projective). Since Theorem 2 concerns a simple case which can be proven and stated without reference to derived non-additive functors, we give a direct proof.

Proof of Theorem 2. We need to show that the total complex associated to the double complex (2) is exact. Choose a splitting $Z \rightarrow Y$ of the short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$. In particular, we obtain for every n an inclusion

$$F_n(X, \dots, X, X) \oplus F_n(X, \dots, X, Z) \hookrightarrow F_n(X, \dots, X, Y),$$

and by Proposition 1 and 2, the cokernel is $F_{n+1}(X, \dots, X, X, Z)$. We use these inclusions to produce an increasing filtration Fil_\bullet on the double complex (2) by letting Fil_n be the sub-double complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & F_n(X, \dots, X) \oplus F_n(X, \dots, Z) & \longrightarrow & F_{n-1}(X, \dots, Y) \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & F_n(X, \dots, X) & \longrightarrow & F_{n-1}(X, \dots, X) \longrightarrow \cdots \end{array}$$

The intermediate quotient $\text{gr}_n = \text{Fil}_n / \text{Fil}_{n-1}$ takes the form

$$\begin{array}{ccc} F_n(X, \dots, X) \oplus F_n(X, \dots, Z) & \longrightarrow & F_n(X, \dots, Z) \\ \uparrow & & \\ F_n(X, \dots, X) & & \end{array}$$

where one computes that the maps are the obvious inclusion and projection maps. In particular, the graded quotients of the total complex are exact and hence the total complex itself is exact. \square

4. A PRESENTATION OF THE GROTHENDIECK GROUP

Let \mathcal{A}_0 be a weak Serre subcategory of \mathcal{A} . This implies that the category $\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$ of chain complexes X_\bullet in \mathcal{A} satisfying

- (1) $X_i = 0$ for all $i < 0$,
- (2) $H_i(X_\bullet) = 0$ for all $i \gg 0$,
- (3) $H_i(X_\bullet) \in \mathcal{A}_0$ for all i

is an abelian subcategory of $\text{Ch } \mathcal{A}$. Assume that \mathcal{A} has enough projectives. Let $\mathcal{P} \subset \mathcal{A}$ be the full additive subcategory consisting of all projectives. Let $\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}$ be the full subcategory consisting of those $P_\bullet \in \text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$ such that $P_i \in \mathcal{P}$ for all i . Again, every $P_\bullet \in \text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$ has an Euler characteristic $\chi(P_\bullet) \in K_0(\mathcal{A}_0)$.

Let \sim be the equivalence relation on (the set of isomorphism classes of) $\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}$ generated by the relations:

- (1) $P_\bullet \sim Q_\bullet$ if P_\bullet and Q_\bullet are homotopy equivalent;
- (2) $P_\bullet \sim Q_\bullet$ if there exist short exact sequences $0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow P_\bullet \rightarrow 0$ and $0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow Q_\bullet \rightarrow 0$ in $\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}$.

Note that, since the complexes consist of projective objects, quasi-isomorphic complexes are always homotopy equivalent. Clearly if $P_\bullet \sim Q_\bullet$ then $\chi(P_\bullet) = \chi(Q_\bullet)$ in $K_0(\mathcal{A}_0)$.

Proposition 3. *The map $\chi: (\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}) / \sim \rightarrow K_0(\mathcal{A}_0)$ is a bijection.*

Proof. We denote the mapping cone of a morphism $X_\bullet \rightarrow Y_\bullet$ by $\text{cone}(X_\bullet \rightarrow Y_\bullet)$ and note that there is a short exact sequence

$$0 \longrightarrow Y_\bullet \xrightarrow{\alpha} \text{cone}(X_\bullet \rightarrow Y_\bullet) \xrightarrow{\beta} X_\bullet[-1] \longrightarrow 0.$$

Let $\mathcal{S} = (\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}) / \sim$ be the set of equivalence classes, and let us denote the equivalence class of an X_\bullet by $\{X_\bullet\} \in \mathcal{S}$. The operation

$$\{X_\bullet\} + \{Y_\bullet\} := \{X_\bullet \oplus Y_\bullet\}$$

is well-defined and makes \mathcal{S} into a monoid.

We claim that \mathcal{S} is even a group. Indeed, comparing the short exact sequences

$$0 \longrightarrow X_\bullet \longrightarrow \text{cone}(X_\bullet \xrightarrow{\text{id}} X_\bullet) \oplus X_\bullet \longrightarrow \text{cone}(X_\bullet \xrightarrow{\text{id}} X_\bullet) \longrightarrow 0$$

and

$$0 \longrightarrow X_\bullet \xrightarrow{(\alpha, 0)} \text{cone}(X_\bullet \xrightarrow{\text{id}} X_\bullet) \oplus X_\bullet \xrightarrow{\beta \oplus \text{id}} X_\bullet[-1] \oplus X_\bullet \longrightarrow 0$$

we see that $X_\bullet \oplus X_\bullet[-1] \sim \text{cone}(X_\bullet \xrightarrow{\text{id}} X_\bullet)$, and since $\text{cone}(X_\bullet \xrightarrow{\text{id}} X_\bullet)$ is homotopy-equivalent to 0 we find $\{X_\bullet\} + \{X_\bullet[-1]\} = 0$ in \mathcal{S} .

Now if $0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow Z_\bullet \rightarrow 0$ is a short exact sequence, then we claim that $\{Y_\bullet\} = \{X_\bullet\} + \{Z_\bullet\}$. Indeed, we have a quasi-isomorphism $\gamma: X_\bullet[-1] \rightarrow \text{cone}(Y_\bullet \rightarrow Z_\bullet)$, and comparing the exact sequences

$$0 \rightarrow X_\bullet[-1] \oplus Z_\bullet \xrightarrow{\begin{pmatrix} \text{id} & \gamma \\ 0 & \alpha \end{pmatrix}} X_\bullet[-1] \oplus \text{cone}(Y_\bullet \rightarrow Z_\bullet) \rightarrow Z_\bullet[-1] \oplus X_\bullet[-1] \rightarrow 0$$

and

$$0 \rightarrow Z_\bullet \oplus X_\bullet[-1] \xrightarrow{\begin{pmatrix} \text{id} & 0 \\ 0 & \alpha \end{pmatrix}} X_\bullet[-1] \oplus \text{cone}(Y_\bullet \rightarrow Z_\bullet) \rightarrow Y_\bullet[-1] \rightarrow 0$$

shows that $Y_\bullet[-1] \sim X_\bullet[-1] \oplus Z_\bullet[-1]$ and hence $\{Y_\bullet\} = \{X_\bullet\} + \{Z_\bullet\}$ in \mathcal{S} .

Taking projective resolutions of objects in \mathcal{A}_0 defines an injective homomorphism $\psi: K_0(\mathcal{A}_0) \rightarrow \mathcal{S}$. To see that it is surjective, we use induction on the amplitude of a complex. We say that X_\bullet has amplitude $\leq a$ if there is an n so that $H_i(X_\bullet) = 0$ for all $i < n$ and $i \geq n + a$. If $X_\bullet \in \text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{P}$ has amplitude ≤ 1 then up to shift X_\bullet is a projective resolution of an object in \mathcal{A}_0 , and lies in the image of ψ . If $a \geq 2$ and X_\bullet has amplitude $\leq a$, then for suitable m the “good truncation” $\tau_{<m} X_\bullet$ (with homology in \mathcal{A}_0 , but necessarily consisting of projectives) gives a short exact sequence

$$0 \rightarrow \tau_{<m} X_\bullet \rightarrow X_\bullet \rightarrow \tau_{\geq m} X_\bullet \rightarrow 0$$

in $\text{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$ with $\tau_{<m} X_\bullet$ and $\tau_{\geq m} X_\bullet$ of amplitude $\leq a - 1$. This sequence is quasi-isomorphic with a short exact sequence

$$0 \rightarrow U_\bullet \rightarrow X'_\bullet \rightarrow V_\bullet \rightarrow 0$$

consisting of complexes of projectives. Since X_\bullet and X'_\bullet consist of projectives they are homotopy-equivalent, and since U_\bullet and V_\bullet have amplitude $\leq a - 1$ we conclude that $\{X_\bullet\} = \{U_\bullet\} + \{V_\bullet\}$ lies in the image of ψ . \square

5. DERIVED FUNCTORS OF NON-ADDITIVE FUNCTORS

For an abelian category \mathcal{A} we denote by $\text{Simp } \mathcal{A}$ the category of simplicial objects in \mathcal{A} and by $\text{HoSimp } \mathcal{A}$ its homotopy category (whose objects are the objects of $\text{Simp } \mathcal{A}$, and whose morphisms are the homotopy classes of morphisms). Any functor $F: \mathcal{A} \rightarrow \mathcal{B}$ induces a functor $\text{Simp } \mathcal{A} \rightarrow \text{Simp } \mathcal{B}$ which is compatible with simplicial homotopy, and hence induces a functor $\text{HoSimp } \mathcal{A} \rightarrow \text{HoSimp } \mathcal{B}$.

A simplicial object $X_\bullet \in \text{Simp } \mathcal{A}$ gives a chain complex

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow 0$$

in the usual way, and this induces a functor $C: \text{HoSimp } \mathcal{A} \rightarrow \mathcal{K}_{\geq 0} \mathcal{A}$. We will use the following variant of the Dold-Kan theorem.

Theorem 3. *The functor $C: \text{HoSimp } \mathcal{A} \rightarrow \mathcal{K}_{\geq 0} \mathcal{A}$ is an equivalence of categories.*

Proof. This is the Dold-Kan theorem [9, 8.4.1], except that we use the full associated chain complex $C(X_\bullet)$ instead of the normalized complex $N(X_\bullet)$. It is not hard to show that the inclusion $N(X_\bullet) \subset C(X_\bullet)$ is a homotopy equivalence. See also [5, III.2.4]. \square

Let $\mathcal{P} \subset \mathcal{A}$ be the additive subcategory of projectives, and assume \mathcal{A} has enough projectives. By the Dold-Kan theorem for every $X_\bullet \in \text{Ch}_{\geq 0} \mathcal{A}$ there exists a $P_\bullet \in \text{Simp } \mathcal{P}$ with a quasi-isomorphism $\alpha: s(P_\bullet) \rightarrow X_\bullet$, and (P_\bullet, α) is unique up to

a unique simplicial homotopy equivalence. This construction defines a ‘simplicial projective resolution’ functor

$$\rho: \mathcal{K}_{\geq 0}\mathcal{A} \rightarrow \mathrm{HoSimp} \mathcal{P},$$

which is the essential ingredient in the definition of derived functors of non-additive functors.

Definition 3. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Then the composition

$$\mathcal{K}_{\geq 0}\mathcal{A} \xrightarrow{\rho} \mathrm{HoSimp} \mathcal{P} \xrightarrow{F} \mathrm{HoSimp} \mathcal{B} \xrightarrow{s} \mathcal{K}_{\geq 0}\mathcal{B}$$

is called the *total derived functor* of F , and denoted LF .

Finally, let $\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P}$ be the full subcategory of $\mathrm{Simp} \mathcal{P}$ consisting of those simplicial objects P_\bullet with bounded homology contained in \mathcal{A}_0 . Let \sim be the equivalence relation on $\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P}$ generated by

- (1) $P_\bullet \sim Q_\bullet$ if P_\bullet and Q_\bullet are homotopy equivalent,
- (2) $P_\bullet \sim Q_\bullet$ if there exist short exact sequences $0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow P_\bullet \rightarrow 0$ and $0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow Q_\bullet \rightarrow 0$ in $\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P}$.

Corollary 1. *The map $\chi: (\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P})/\sim \rightarrow K_0(\mathcal{A}_0)$ is a bijection.*

Proof. This follows from Proposition 3 and Theorem 3. \square

6. PROOF OF THE MAIN RESULT

Let \mathcal{A}_0 be a weak Serre subcategory of \mathcal{A} . Assume that every $X \in \mathcal{A}_0$ has a projective resolution in \mathcal{A} . Let $\mathcal{P} \subset \mathcal{A}$ be the exact category of all projectives. Let $\mathrm{Simp}^{\mathcal{B}_0} \mathcal{B}$ be the full subcategory of $\mathrm{Simp} \mathcal{B}$ consisting of those simplicial objects with bounded homology contained in \mathcal{B}_0 .

Theorem 4. *Let d be a positive integer. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor with $F(0) = 0$. Assume F is of degree $\leq d$ and that it maps $\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P}$ to $\mathrm{Simp}^{\mathcal{B}_0} \mathcal{B}$. Then there exists a unique map $f: K_0(\mathcal{A}_0) \rightarrow K_0(\mathcal{B}_0)$ such that the square*

$$\begin{array}{ccc} \mathrm{Simp}^{\mathcal{A}_0} \mathcal{P} & \xrightarrow{F} & \mathrm{Simp}^{\mathcal{B}_0} \mathcal{B} \\ \downarrow \chi & & \downarrow \chi \\ K_0(\mathcal{A}_0) & \xrightarrow{f} & K_0(\mathcal{B}_0) \end{array}$$

commutes. Moreover, the map f is of degree $\leq d$.

This theorem implies Theorem 1 of the introduction.

Proof. By Corollary 1, the map f is unique, and to establish existence it suffices to show that for every P_\bullet, Q_\bullet in $\mathrm{Simp}^{\mathcal{A}_0} \mathcal{P}$ we have that $P_\bullet \sim Q_\bullet$ implies $\chi(F(P_\bullet)) = \chi(F(Q_\bullet))$ in $K_0(\mathcal{B}_0)$. Recall that the equivalence relation \sim is generated by homotopy equivalences and by relations coming from short exact sequences.

If P_\bullet and Q_\bullet are simplicially homotopy-equivalent, then $F(P_\bullet)$ and $F(Q_\bullet)$ are simplicially homotopy-equivalent, and hence $\chi(F(P_\bullet)) = \chi(F(Q_\bullet))$ in $K_0(\mathcal{B}_0)$.

Now let

$$0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow P_\bullet \rightarrow 0$$

be a short exact sequence in $\text{Simp}^{A_0} \mathcal{P}$. Note that for every n the sequence $0 \rightarrow X_n \rightarrow Y_n \rightarrow P_n \rightarrow 0$ is split (since P_n is projective), but that the sequence in $\text{Simp}^{A_0} \mathcal{P}$ need not split. For every n the construction of §3 gives a double complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_3(X_n, X_n, Y_n) & \longrightarrow & F_2(X_n, Y_n) & \longrightarrow & F(Y_n) \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F_3(X_n, X_n, X_n) & \longrightarrow & F_2(X_n, X_n) & \longrightarrow & F(X_n) \end{array}$$

whose total complex is a resolution of $F(P_n)$. Since the construction of the double complex is functorial in the map $X \rightarrow Y$, we obtain a double complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_3(X_\bullet, X_\bullet, Y_\bullet) & \longrightarrow & F_2(X_\bullet, Y_\bullet) & \longrightarrow & F(Y_\bullet) \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F_3(X_\bullet, X_\bullet, X_\bullet) & \longrightarrow & F_2(X_\bullet, X_\bullet) & \longrightarrow & F(X_\bullet) \end{array}$$

in $\text{Simp} \mathcal{B}$, whose associated total complex is a resolution of $F(P_\bullet)$. Because F is of finite degree, this is a *finite* resolution.

Each of the terms is a direct summand of a simplicial object of the form

$$F(X_\bullet \oplus \cdots \oplus X_\bullet) \quad \text{or} \quad F(X_\bullet \oplus \cdots \oplus X_\bullet \oplus Y_\bullet),$$

and hence lies in $\text{Simp}^{\mathcal{B}_0} \mathcal{B}$. This means that in $K_0(\mathcal{B}_0)$ we have

$$\chi(F(P_\bullet)) = \sum_{n=1}^d (-1)^n \left(\chi(F_n(X_\bullet, \dots, X_\bullet)) - \chi(F_n(X_\bullet, \dots, Y_\bullet)) \right).$$

In particular, since the terms do not depend on the map $X_\bullet \rightarrow Y_\bullet$, we see that if

$$0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow Q_\bullet \rightarrow 0$$

is a second short exact sequence in $\text{Simp}^{A_0} \mathcal{P}$ then $\chi(F(P_\bullet)) = \chi(F(Q_\bullet))$ in $K_0(\mathcal{B}_0)$. This proves the existence of f .

Finally, note that the $(d+1)$ -st cross effect of the functor

$$F: \text{Simp}^{A_0} \mathcal{P} \rightarrow \text{Simp}^{\mathcal{B}_0} \mathcal{B}$$

vanishes, which implies the analogous statement for the function f , and shows that f is of degree $\leq d$. \square

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